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The Illumination of Polygonal Regions with Modems

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Abstract

In this paper, the problem of finding the minimum number of k-modems to cover a monotone polygon is considered. A k-*modem* is a point guard which can see a point, if the line segment joining them crosses at most k edges of the polygon. The parameter k is referred as the *power* of the k-modem. It is shown that every monotone polygon on n vertices can be illuminated with $\left\lceil \frac{n}{2k+2} \right\rceil$ k modems. This bound is tight. It is also shown that every orthogonal polygon (with or without holes) on $2m$ vertices can be covered with a $(m - 4)$- modem for odd m and with a $(m - 3)$- modem for even m. When the purpose is covering a simple orthogonal polygon with a single modem placed at a point in its interior or boundary, these bounds on the power of the modem are tight. It is also shown if there is a monotone orthogonal polygon in the plane, a 4-modem will be enough to illuminate the plane. This bound on the power of the modem is tight.

Keywords: Illumination, K-Modem, Visibility, Art gallery, Computational Geometry.

1. Introduction

The notion of modem visibility was introduced by aichholzer et al. [1] in context of the art gallery problem. They studied the k -*modem art gallery problem*. The problem is to determine the number of k-modems sufficient and necessary to cover (or, illuminate) every point in the interior of a gallery modeled by a polygon. A k-*modem* is a point guard which can see (or cover, or illuminate) a point, if the line segment joining them crosses at most k edges of the polygon. So, the k-modem art gallery problem for $k = 0$ corresponds to the classical art gallery problem which was studied by researchers in the field of computational geometry [2, 3]. Throughout our paper, we can use the terms "covering", "illumination", and "guarding" interchangeably because they have the same meaning in this context. The main motivation of that research arises, when you are trying to connect your laptop to a wireless modem. The distance from your laptop to the wireless modem and the number of walls separating them are two crucial factors to have a stable connection in order to navigate the Web. It naturally seems that in most buildings, the number of walls is more important. As a matter of fact, we assume that all the walls are made from homogeneous materials and have the same thickness.

Aichholzer et al. [1] defined a k-modem which is a wireless modem strong enough to transmit a stable signal through at most k walls. They studied the k-modem art gallery problem for monotone polygons and showed that every monotone polygon on n vertices can be covered with $\left\lceil \frac{n}{2k} \right\rceil$ k- modems, and they exhibited examples of mono tone polygons requiring $\left\lceil \frac{n}{2k+2} \right\rceil$ k - modems. For monotone orthogonal polygons, they showed that every such polygon on n vertices can be illuminated with $\left\lceil \frac{n}{2k+4} \right\rceil$ k- modems and give examples which require $\left\lceil \frac{n}{2k+4} \right\rceil$ k- modems for even k and $\left\lceil \frac{n}{2k+6} \right\rceil$ k- modems for odd k.

Fabila-Monroy et al. [4] introduced a new problem in the modem visibility field which was *the* k*-modem covering the plane*. This problem can be stated as follows: Given a polygon, the problem is to determine the number of k-

modems sufficient and necessary to cover every point in the plane so that the modems can be located in the exterior or interior of the polygon. They showed that the plane in presence of an orthogonal polygon on $2m$ vertices can be covered with a single $(m-1)$- modem for even m and with a single m-modem for odd m. This modem is located in the interior of the polygon. They also showed that the whole plane in the presence of a simple polygon or an orthogonal polygon P on n vertices can always be covered with a single $\left\lceil \frac{2n+1}{3} \right\rceil$- modem or a single $\left\lceil \frac{n}{3} \right\rceil$- modem, respectively. The proofs of theorems in [4] show that these modems may be placed in the exterior of the polygon.

Ballinger et al. [5] gave some lower and upper bounds for the number of k-modems necessary and sufficient to cover a given collection of line segments, polygonal chains and polygons. They used the word of transmitter instead of modem.

In this paper, it is shown that every monotone polygon on n vertices can be illuminated with $\left\lceil \frac{n}{2k+2} \right\rceil$ k- modems. Since Aichhlozer et al. [1] exhibited examples of monotone polygons requiring $\left\lceil \frac{n}{2k+2} \right\rceil$ k- modems, our bound is tight. It is also shown that every orthogonal polygon on $2m$ vertices (with or without holes) can be covered with a single $(m-4)$-modem for odd m and with a single $(m-3)$- modem for even m. When the purpose is covering a simple orthogonal polygon with a single modem placed at a point in its interior or boundary, these bounds on the power of the modem are tight. The results so far are related to the cover a polygon in the plane. They do not consider the cover the whole plane. In contrast, the following result is related to the cover the whole plane in the presence of a polygon as an obstacle. It is shown that if there is a monotone orthogonal polygon in the plane, a single 4-modem will be enough to illuminate the plane. This bound on the power of the modem is tight.

This paper is organized as follows. Section 2 gives the upper bound $\left\lceil \frac{n}{2k+2} \right\rceil$ on the minimum numbers of required k-modems to cover a monotone polygon and we show this bound is tight. In section 3 we show that every orthogonal polygon (with or without holes) on $2m$ vertices can be covered with a $(m-4)$-modem for odd m and with a $(m-3)$-modem for even m. In section 4 we discuss about the problem of k- modem covering the plane. In section 5 we conclude some remarks and open problems.

2. The k-Modem Art Gallery Problem for Monotone Polygons

In this section, we want to deal with the k-modem art gallery problem for monotone polygons. Our goal is finding the number of k-modems which are always sufficient and sometimes necessary for covering an arbitrary monotone polygon.

If k-modems are placed at vertices of a polygon, they are called vertex k-modems. If a modem is located at a point of an edge, this edge will not block any signals of the modem. If it is located at a vertex, the incident edges will not block.

Lemma 1. Every $(k+3)$-gon can be covered with a vertex k-modem.

Proof. This is straightforward because every line segment joining a vertex and a point in the interior or on the boundary of a $(k+3)$-gon crosses at most k edges of the polygon. ∎

Lemma 2. Every monotone polygon on $2k+4$ vertices can be illuminated with a single k-modem placed on the boundary of the polygon, and there is a monotone polygon on $2k+6$ vertices which cannot be illuminated with a single vertex k-modem.

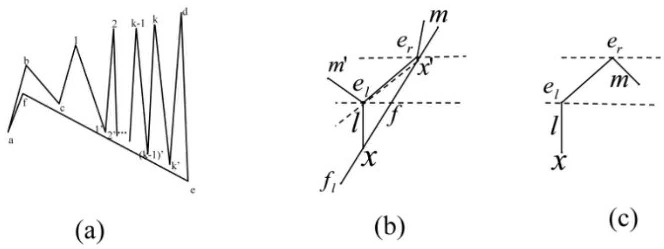

Figure 1. the proof of lemma 2. **(a)** An example of monotone polygon with $2k+6$ vertices which cannot be illuminated with a single vertex k-modem, **(b)** some parts of the polygon which are adjacent to the edge e. A k-modem placed at the point x′ covers the whole polygon, **(c)** A k-modem placed at the point x covers the whole polygon

Proof. An example of monotone polygon with $2k+6$ vertices which cannot be illuminated with a single vertex k-modem is shown in Figure 1(a). As shown in this figure, the vertices labeled by a, b, c, d, e, and f are fixed for every positive value of the parameter k. When the value of k increases, the number of intermediate peaks increases. As the polygon is x-monotone, we can sort its vertices from left to right according to their x coordinate.

The height or y-coordinate of convex vertices (except the vertices a, e, f) increase monotonically from left to right. Likewise, The height or y-coordinate of reflex vertices (except the vertex labeled by $\left\lceil \frac{k}{2} \right\rceil'$) decrease monotonically from left to right. The convex vertices of the polygon are denoted by $1, 2, \ldots, k$, and its reflex vertices are denoted by $1', 2', \ldots, k'$, as shown in Figure 1(a).

The polygon has some special properties as follows:

1) For every pair of a convex and a reflex vertex, consider the line segment joining them. All the convex (reflex) vertices between them lie above (below) this line segment.

2) All the reflex vertices which lie to the left (right) of the vertex $\left\lceil \frac{k}{2} \right\rceil$, and the vertex a (e) lie below the line segment joining the vertices $\left\lceil \frac{k}{2} \right\rceil$ and f (k′).

3) All the convex vertices which are to the left (right) of the vertex $\left\lceil \frac{k}{2} \right\rceil$ lie above the line segment joining the vertices $\left\lceil \frac{k}{2} \right\rceil$ and f (k′).

4) All the reflex vertices which lie to the left (right) of the vertex $\left\lceil \frac{k}{2} \right\rceil'$, and the vertex a (e) lie below the line segment joining the vertices $\left\lceil \frac{k}{2} \right\rceil'$ and f (d).

5) All the convex vertices which lie to the left (right) of the vertex $\left\lceil\frac{k}{2}\right\rceil'$ lie above the line segment joining the vertices $\left\lceil\frac{k}{2}\right\rceil'$ and f (d).

According to the property 1, it is obvious that the only candidates for covering this polygon with a single vertex k-modem are the k-modem placed at the vertex $\left\lceil\frac{k}{2}\right\rceil$ for odd k, and the k-modem placed at the vertex $\left\lceil\frac{k}{2}\right\rceil'$ for even k. However, According to the other properties mentioned above, there is at least a point on the polygon which do not cover with these k-modems for each case.

Without loss of generality, we suppose that P is an x-monotone polygon with vertices $p_1, p_2, \cdots, p_{2k+4}$ ordered from left to right. Without loss of generality, we assume that p_{k+2} lies on the upper polygonal chain of P. Imagine that an edge of P having p_{k+2} as its left endpoint is denoted by e. Thus let e_l and e_r be the left and right endpoints of e, respectively. Assume that e_r and p_{2k+4} are not the same. So, there is an adjacent edge to e_l denoted by m′ and the other adjacent edge to e_r denoted by m. The edges m and m′ are on the upper polygonal chain of P.

Draw a vertical line through p_{k+2}. Let x be the intersection point of this vertical line and the boundary of P. The vertical line segment joining p_{k+2} and x is denoted by l. Let P_1^- be the subset of P to the left of l. Let P_1^+ be the subset of P to the right of l. We consider $L = P_1^- \cup l$ and $R = P_1^+ \cup l$.

If x is a vertex of P, then L and R have $k + 3$ vertices. According to Lemma 1, a k-modem placed at x or p_{k+2}, can cover L and R. So, it can cover L either. Now we suppose that x is not a vertex of P. So, the edge of P lying directly below P_{x+2} is denoted by f with f_l and f_r as its left and right endpoints. The polygons L and R have $k + 3$ and $k + 4$ vertices, respectively. We have two situations:

1. e_r lies above the horizontal line passing through e_l.Two cases occur.

A. The right endpoint of m lies above the horizontal line through e_r, see Figure 1(b).

- The point x lies below the intersection point of the extensions of l and m. So, There is no ray emanated from x which can cross both of e and m. So, the line segment joining x and any point of R crosses at most k edges of R. Therefore, a k-modem placed at x, can cover R. Since x is a vertex of L having $k + 3$ edges, a k-modem placed at x can cover L either, according to Lemma 1. So, it can cover all the points of P.
- The point x lies above or on the intersection point of the extensions of l and m.
  - ✓ There is no rays emanated from p_{k+2} which can cross both of f and m simultaneously. So, the line segment joining p_{k+2} and any point of R can cross at most k edges of R. As before, a k-modem placed at p_{k+2}, can cover P.
  - ✓ There is at least a ray emanated from p_{k+2} that crosses both of f and m simultaneously. So, the vertex f_r should lie above the horizontal line through e_r. It is obvious that e_r and p_{k+3} are the same. On the other hand, the intersection point of the extension of e and the edge f should lie to the right of the line containing l.

Draw a vertical line through p_{k+3}. Let x′ be the intersection point of this vertical line and the boundary of P. The vertical line segment joining p_{k+3} and x′ is denoted by l′. Let P_2^- be the subset of P to the left of l′. Let P_2^+ be the subset of P to the right of l′. We consider $L' = P_2^- \cup l'$ and $R' = P_2^+ \cup l'$. If x′ is a vertex of P, as before, L′ and R′ have $k + 3$ vertices. According to Lemma 1, a k-modem placed at x′ or p_{k+3} can cover L and R. So, this modem can cover P.

Now we suppose that x′ is not a vertex of P. So, the point x′ lies on the edge f. Therefore, R′ has $k + 3$ vertices and L′ has $k + 4$ vertices. We have two situations.

- o There is no ray from x′ crossing both of e and m′ simultaneously. The line segment joining x′ and any point of L′ crosses at most k edges of L′ simultaneously. Therefore, a k-modem placed at x′ covers L′. Since x′ is a vertex of R′ having $k + 3$ edges, a k-modem placed at x′ also covers R′, according to Lemma 1. So, this modem covers P.
- o There is at least one ray emanated from x′ crossing both of e and m′. Therefore, the left endpoint of m′ should lie above the line connecting x′ and e_l. Since x′ and p_{k+3} lie on the same vertical line and x′ lies below p_{k+3}, the left endpoint of m′ should lie above the extension of the edge e. So, there is no ray emanated from p_{k+3} crossing both of f and m′ simultaneously. Therefore, a k-modem placed at p_{k+3}, can cover L′. As before, this modem can cover P.

B. The right endpoint of m lies below the horizontal line passing through e_r, see Figure 1(c). In this case, there is no ray emanated from x crossing both of e and m at the same time because the polygon P is x-monotone. So, the line segment joining x and any point of R crosses at most k edges of R. Since x is a vertex of L having $k + 3$ edges, a k-modem placed at x also covers L, according to Lemma 1. So, this modem covers P.

2. e_r lies below the horizontal line passing through e_l. Two cases occur.

A. The left endpoint of m lies above the horizontal line passing through e_r.

B. The left endpoint of m lies below the horizontal line passing through e_r.

These cases are similar to the case of 1A. Analogous arguments show that a k-modem placed at x or x′ or p_{k+2} or p_{k+3} can cover all the points of P.

C. If e_r and p_{2k+4} are the same, we can argue on the edges being adjacent to f in a similar way. ■

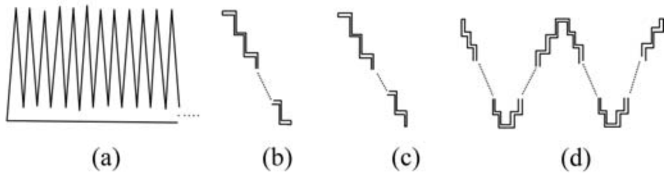

Figure 2. **(a)** The monotone polygons on n vertices which require $\lceil n/(2k + 2)\rceil$ k-modems to be covered [1], **(b)**, **(c)** the orthogonal 2m-gons (without hole) that being illuminated require a single $(m - 4)$-modem for odd m and a single $(m - 3)$-modem for even m, **(d)** The lower bound on the power of a single modem to cover the whole plane in presence of the monotone orthogonal polygon is 4

Lemma 3. (Splitting Lemma) [1] Let P be an x-monotone polygon with vertices $p_1, p_2, \cdots, p_n$, ordered from left to right. For every positive integer $m < n$, there exist a vertical line segment l and two monotone polygons L and R, such that:

- L has m vertices and R has $n - m + 2$ vertices.
- Either l is chord of L and an edge of R, or l is an edge of L and a chord of R.
- p_m or p_{m+1} is an end point of l.
- Denote as L' the subset of L to the left of l, and denote as R' the subset of R to the right of l; then $P = L' \cup R' \cup l$.

Theorem 1. Every x-monotone n-gon can be illuminated with $\left\lceil \frac{n}{2k+2} \right\rceil$ k-modems. This bound is tight.

Proof. As before, we suppose that P is an x-monotone polygon with vertices $p_1, p_2, \cdots, p_n$, ordered from left to right. We apply Lemma 3 to P and obtain two x-monotone polygons with $2k + 4$ and $n - 2k - 2$ vertices are denoted by L and R, respectively. Again, we apply Lemma 3 to R and obtain two x-monotone polygons with $2k + 4$ and $n - 4k - 4$ vertices are denoted by L' and R', respectively. We continue this process. Finally, according to Lemma 3, we obtain $\left\lceil \frac{n}{2k+2} \right\rceil$ monotone polygons with at most $2k + 4$ vertices covering P. According to Lemma 2, any of these polygons can be covered with a k-modem. Therefore, P can be covered with $\left\lceil \frac{n}{2k+2} \right\rceil$ k-modems. Aichhlozer et al. [1] exhibited examples of monotone polygons requiring $\left\lceil \frac{n}{2k+2} \right\rceil$ k-modems shown in Figure 2(a). So, the upper bound on the minimum number of required k-Modems to cover a monotone polygon is tight. ■

3. The k-Modem Art Gallery Problem for Orthogonal Polygons

In this section, the k-modem art gallery problem for orthogonal polygons is considered. The purpose is finding the lower bounds on the power of a single modem, placed in the interior of an orthogonal polygon to cover it.

Definition 1. given a point $p = (a, b)$ in the plane. Let $C_+^+(p) = \{(x, y) | x \geq a, y \geq b\}$, $C_+^-(p) = \{(x, y) | x \geq a, y \leq b\}$, $C_-^+(p) = \{(x, y) | x \leq a, y \geq b\}$ and $C_-^-(p) = \{(x, y) | x \leq a, y \leq b\}$.

Lemma 4. [4] Let P be an orthogonal polygon and p be a point in the plane. Suppose that k vertices of P belong to the interior of $C_+^+(p)$, then any ray emanated from p, contained in $C_+^+(p)$, crosses at most k edges of P. This statement is also true for $C_-^+(p)$, $C_+^-(p)$ and $C_-^-(p)$.

In Lemma 5, the purpose is covering all points of the plane. If the goal is covering only the interior of a polygon, then we have the following corollary.

Corollary 1. Let P be an orthogonal polygon and p be a point in the plane. Consider m vertices of P belong to the interior of $C_+^+(p)$, then the line segment joining p and any point in the interior of P and $C_+^+(p)$, crosses at most $m - 1$ edges of P. (This statement is also true for $C_-^+(p)$, $C_+^-(p)$ and $C_-^-(p)$.)

Theorem 2. Every orthogonal polygon in general position (with or without holes) on 2m vertices can be illuminated with a single $(m - 4)$-modem for odd m and with a single $(m - 3)$-modem for even m. When the goal is to cover only the interior of a simple orthogonal polygon with a single modem placed at a point in its interior or boundary, these bounds on the power of the modem are tight.

Proof. Let m be odd. There is a horizontal line containing a horizontal edge of P so that $\frac{m-1}{2}$ horizontal edges of P lie above and the remaining lie below it. Let L and e denote this line and this edge, respectively. Likewise e' and e'' denote the left and the right endpoints of e, respectively. We assume that the polygon is in a general position, i.e. none of the two edges of P is collinear.

Since we should have a polygon, L would cross a vertical edge of P which its upper endpoint lies above L and its other endpoint lies below it. If there is more than one vertical line intersecting e, we consider the rightmost one. Assume f and M to be this vertical edge and its extension, respectively. The intersection point of L and M lies on the polygon boundary. Let g and k denote the first horizontal edges of P lying below and above e, respectively. Assume G and K to be the extensions of them, respectively. Without loss of generality, we suppose that ≥ 4 . Three cases occur:

1. Any of the four regions defined by L and M contains at least one vertex of the polygon. There are at most $m - 3$ vertices in the interior of any of these regions. According to Corollary 1, the line segment joining the intersection point of L with M and any point in the interior of the polygon crosses at most $m - 4$ edges. So, a single $(m - 4)$-modem placed at the intersection point of L and M can illuminate the interior of the polygon.

2. Just two regions of the whole regions defined by L and M contain no vertex of the polygon. So, the edge f is the rightmost or the leftmost vertical edge of P. If there is not a vertical edge (expect the edge f) such that its upper and lower endpoints lie above and below L, respectively, the edge f is the leftmost vertical edge of P. Firstly, we assume that f is the rightmost vertical edge of P. So, two regions which are to the right of M are empty. We order the vertical edges except f, from right to left. Suppose that the first and second vertical edges in the ordered list are denoted by f_1 and f_2 respectively. As we suppose that $m \geq 4$, these vertical edges actually exist. To clarify, we have following states:

- If the upper endpoint of the f_1 lies above L and its lower endpoint lies below L, a single $m - 4$-modem is placed at the intersection point of L and f_1. This modem covers the polygon because there are at most $m - 3$ vertices in the interior of any regions defined by L and the extension of f_1.
- if the upper endpoint of f_1 lies below L, consider f_2.
  - ✓ If the upper endpoint of the f_2 lies above L and its lower endpoint lies below L, a single $(m - 4)$-modem is placed at the intersection point of L and f_2. This modem covers the polygon because there are at most $m - 3$ vertices in the interior of any regions defined by L and the extension of f_2.

- ✓ If the upper endpoint of f_2 lies below L, a single (m – 4)-modem is placed at the intersection point of K and the extension of f_2. This intersection point lies inside the polygon. If this point does not lie inside the polygon, there should be a vertical edge whose lower endpoint lies below L and upper endpoint lies above K which is a contradiction. Because we assume that the upper endpoint of f_1 lies below L. This modem covers the polygon because there are at most m – 3 vertices in the interior of any regions defined by K and the extension of f_2.
- ✓ If the lower endpoint of f_2 lies above, a single (m – 4)-modem is placed at the intersection point of L and the extension ot f_2. This modem covers the polygon because there are at most m – 3 vertices in the interior of any regions defined by L and the extension of f_2. This intersection point lies inside the polygon. If this point does not lie inside the polygon, there should be a vertical edge whose lower endpoint lies below L and upper endpoint lies above L which is a contradiction. Because we assume that the upper endpoint of f_1 lies below L.

- The lower endpoint of f_1 lies above L. The arguments in this case are similar to the above case.
- The lower endpoint of f_1 lies on L. So, f_1 and e have a common end point. Consider f_2.
  - ✓ If the lower endpoint of the f_2 lies above L, a single (m – 4) -modem is located at the intersection point of the extension of f_2 and G.
  - ✓ If the upper endpoint of the f_2 lies below L, a single (m – 4)modem is located at the intersection point of the extension of f_2 and L. This point lies on the edge e.
  - ✓ If the lower endpoint of the f_2 lies on L, f_2 and e have a common end point. So, a single (m – 4)-modem is located at the intersection point of the extension of f_2 and G.
  - ✓ If the upper endpoint of the f_2 lies on L, f_2 and e have a common end point. So, a single (m – 4)-modem is located at the intersection point of the extension of f_2 and G.
- The upper endpoint of f_1 lies on L. So, the discussion about this case is similar to the above. Therefore, we ignore it.

If the edge f is the leftmost vertical edge of P, we can argue about it in a very similar way. However, we should sort the vertical line from left to right.

3. Just one region of the whole regions defined by L and M contains no vertex of the polygon. Assume p to be the intersection point of f and L. Without loss of generality, we suppose that $C_+^-(p)$ is empty. So, there are at most m – 4, m – 3, and m – 2 vertices inside $C_-^+(p)$, $C_+^+(p)$, and $C_-^-(p)$, respectively.

Without loss of generality, we assume that the endpoints of e are different from those of f. (If their endpoints are the same, than there is at least one vertical edge having the properties of f and its endpoints and the endpoints of e are not the same.)

Now we suppose that there are at least two vertices inside $C_-^+(p)$. Therefore, there are at most m – 4 vertices inside $C_+^+(p)$. We order the vertical edges that their upper endpoint lies inside $C_-^+(p)$ from right to left. Let us consider the first one. If its lower endpoint lies below L, then a single (m – 4)-modem is placed at the intersection point of L and that vertical edge. Otherwise, a single (m – 4)-modem is placed at the intersection point of G and that vertical edge. This intersection point lies inside the polygon.

If there is one vertex inside $C_-^+(p)$, two situations are observed.

- The x-coordinate of the right endpoint of e is less than that of the intersection point of L and M. A single (m – 4)-modem is placed at the intersection point of M and G. This intersection point lies inside f because g denotes the first horizontal edges of P lying below e.
- The x -coordinate of the left endpoint of e is more than that of the intersection point of L and M.

Since we have a polygon in the general position, i.e. none of the two edges of P is collinear, L crosses another vertical edge of P that its upper endpoint lies above L and its other endpoint lies below L. Assume f′ and M′ to be this vertical edge and its extension, respectively. Lines L and M′ define four regions, as before.

If there is at least one vertex in any of these regions, then, a single (m – 4)-modem is placed at the intersection point of L and M′. if just two regions of the whole regions defined by L and M′ contain no vertex of the polygon. So, f′ is leftmost vertical edge of P. We order the vertical edges that their upper endpoint lies above L except f′, from left to right. Similar to the case 2, the suitable placement can be found for a single (m – 4)-modem.

Now suppose that just one region of the whole regions defined by L and M′ contains no vertex of the polygon. Assume p′ to be the intersection point of f′ and L. Therefore, one of $C_-^+(p')$ and $C_-^-(p')$ is empty.

Firstly, we suppose that $C_-^+(p')$ is empty. So, there are at most m – 4, m – 3, and m – 2 vertices inside $C_+^-(p')$, $C_-^-(p')$, and $C_+^+(p')$, respectively.

If there is one vertex inside $C_+^-(p')$, then a single (m – 4)-modem is placed at the intersection point of M′ and K This intersection point lies inside f′ because k denotes the first horizontal edges of P lying above e.

Now we suppose that there are at least two vertices inside $C_+^-(p')$ Therefore, there are at most m – 4 vertices inside $C_-^-(p')$. We order the vertical edges that their lower endpoint lies inside $C_+^-(p')$, from left to right. Let us consider the first one. If its upper endpoint lies above L, then a single (m – 4)-modem is placed at the intersection point of L and that vertical edge. Otherwise, a single (m – 4)- modem is placed at the intersection point of K and the extension of that vertical edge. This intersection point lies inside the polygon because k denotes the first horizontal edges of P lying above e and we consider the first vertical edge in the ordered list.

Secondly, we suppose that $C_-^-(p')$ is empty. So, there are at most m – 2, m – 5, and m – 4 vertices inside $C_+^-(p')$, $C_-^+(p')$, and $C_+^+(p')$, respectively. We order the vertical edges that their lower endpoint lies inside $C_+^-(p')$, from left to right. Let us consider the first one. If its upper endpoint lies above L, then a single (m – 4)-modem is placed at the intersection point of L and that vertical edge. Otherwise, a single (m – 4)-modem is placed at the intersection point of L and the extension of that vertical edge. This intersection point lies inside the polygon because we consider the first vertical edge in the ordered list.

Now let m be even. There exists a horizontal line which contains a horizontal edge of P such that $\frac{m}{2}$ horizontal edges of P lies above and the rest lie below it. As before, let L denote this line. Since we have a polygon, L should cross at least a vertical edge of P.

Assume f and M to be this vertical edge and its extension, respectively. Let g denote the first horizontal edges of P lying below L and G denote the extensions of g. As before, three situations are considered in which the first and second are similar to odd m.

Now suppose that one region of the total regions defined by L and M contains no vertex of the polygon. A modem is placed at the intersection point of G and M. It is easy to see that the intersection point lies inside the f. In the following, it is shown the orthogonal 2m-gons (without hole) that being illuminated require a single $(m-4)$-modem for odd m and a single $(m-3)$-modem for even m. So, it is shown when the purpose is illuminating the interior of the polygon with a single modem placed at a point in the interior or boundary of the polygon, these bounds on the power of the modem are tight. See Figure 2(b) and 2(c). ■

Note that Fabila-Monroy et al. [4] showed that the plane in presence of an orthogonal polygon on 2m vertices can covered with a single $(m-1)$-modem for even m and with a single m-modem for odd m which this modem located in the interior or boundary of polygon. It seems strange that our upper bounds in this theorem are larger for even m, we get one $(m-3)$-modem for even m, and one $(m-4)$-modem for odd m. Whereas the previous result [4] had a larger bound for odd m, they got one $(m-1)$-modem for even m and one m -modem for odd m. Now suppose that m is odd and the goal is covering the whole plane.

In Theorem 2, there exists at most $m-2$ vertices in each regions defined by L and M. According to Lemma 5 [4], any ray emanated from the intersection of L and M crosses at most $m-2$ edges of the polygon for covering the whole plane. So, a single $(m-2)$-modem which is placed at the intersection of L and M can cover the whole plane. Therefore, the plane in presence of an orthogonal polygon on 2m vertices can covered with a single $(m-1)$ -modem for even m and with a single $(m-2)$-modem for odd m which this modem located in the interior or boundary of polygon. Therefore, there is not any reversal for odd/even m.

Theorem 3. Every orthogonal polygon which has r reflex vertices $(r > 0)$ would be illuminated with $\left\lceil \frac{r}{2} \right\rceil$ 2-modems.

Proof. The proof is by induction on r, the number of reflex vertices of the orthogonal polygon. It is obvious that the base of the induction is true. If the orthogonal polygon has one reflex vertex, a single 2-modem is enough for covering the polygon. This modem should be placed at the reflex vertex. If the orthogonal polygon has two reflex vertices, a single 2-modem is enough for covering the polygon. This modem should be placed at the convex vertex being between the two reflex vertices.

We assume that any orthogonal polygon with r′ reflex vertices $(0 < r' < r)$ can be covered with $\left\lceil \frac{r'}{2} \right\rceil$ 2-modems, as the induction assumption. Now, suppose that P is an orthogonal polygon with r reflex vertices. We are going to show that it can be covered with $\left\lceil \frac{r}{2} \right\rceil$ 2-modems. We consider a reflex vertex of the polygon P and continue one of its incident edges to the interior of the polygon so that it would hit boundary at the first time. So, two sub polygons with r_1 and r_2 reflex vertices will be obtained. These polygons are called P_1 and P_2, respectively. It is clear that both of P_1 and P_2 are orthogonal. Therefore we apply the induction assumption on them. It is clear that $r = r_1 + r_2 + 1$. Four cases occur as follows:

- Both of r_1 and r_2 are even. So, there exist the integers k_1 and k_2 such that $r_1 = 2k_1$ and $r_2 = 2k_2$. Based on the induction assumption, $\left\lceil \frac{r_1}{2} \right\rceil = k_1$ and $\left\lceil \frac{r_2}{2} \right\rceil = k_2$ are the numbers of 2-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 2-modems that are sufficient for covering P is $k_1 + k_2$. We want to show that $k_1 + k_2 \le \left\lceil \frac{r}{2} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{2} \right\rceil = \left\lceil \frac{r_1+r_2+1}{2} \right\rceil = \left\lceil \frac{2k_1+2k_2+1}{2} \right\rceil = k_1 + k_2 + 1 > k1 + k2$.,Since the polygon P can be covered with $k_1 + k_2$ 2-modems, it can be covered with $\left\lceil \frac{r}{2} \right\rceil$ 2-modems.
- r_1 is even and r_2 is odd. So, there exist the integers k_1 and k_2 such that $r_1 = 2k_1$ and $r_2 = 2k_2 + 1$. Based on the induction assumption, $\left\lceil \frac{r_1}{2} \right\rceil = k_1$ and $\left\lceil \frac{r_2}{2} \right\rceil = k_2 + 1$ are the numbers of 2-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 2-modems that are sufficient for covering P is $k_1 + k_2 + 1$. We want to show that $k_1 + k_2 + 1 = \left\lceil \frac{r}{2} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{2} \right\rceil = \left\lceil \frac{r_1+r_2+1}{2} \right\rceil = \left\lceil \frac{2k_1+2k_2+2}{2} \right\rceil = k_1 + k_2 + 1$. So, the polygon P can be covered with $\left\lceil \frac{r}{2} \right\rceil$ 2-modems.
- r_1 is odd and r_2 is even. This case is similar to the above case.
- Both of r_1 and r_2 are odd. So, there exist the integers k_1 and k_2 such that $r_1 = 2k_1 + 1$ and $r_2 = 2k_2 + 1$. Based on the induction assumption, $\left\lceil \frac{r_1}{2} \right\rceil = k_1 + 1$ and $\left\lceil \frac{r_2}{2} \right\rceil = k_2 + 1$ are the numbers of 2-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 2-modems that are sufficient for covering P is $k_1 + k_2 + 2$. We want to show that $k_1 + k_2 + 2 = \left\lceil \frac{r}{2} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{2} \right\rceil = \left\lceil \frac{r_1+r_2+1}{2} \right\rceil = \left\lceil \frac{2k_1+2k_2+3}{2} \right\rceil = k_1 + k_2 + 2$. So, the polygon P can be covered with $\left\lceil \frac{r}{2} \right\rceil$ 2-modems.■

Theorem 4. Every orthogonal polygon which has r reflex vertices (r> 0) would be illuminated with $\left\lceil \frac{r}{3} \right\rceil$ 3-modems.

Proof. The proof is similar to the proof of theorem 3. The proof is by induction on r, the number of reflex vertices of the orthogonal polygon. It is obvious that the base of the induction is true. If the orthogonal polygon has one reflex vertex, a single 3 -modem is enough for covering the polygon.

This modem can be placed at the reflex vertex. If the orthogonal polygon has two reflex vertices, a single 3-modem is enough for covering the polygon. This modem should be placed at the convex vertex being between the two reflex vertices. If the orthogonal polygon has three reflex vertices, a single 3-modem is enough for covering the polygon. This modem should be placed at the convex vertex being between the three reflex vertices.

We assume that any orthogonal polygon with r′ reflex vertices ($0 < r' < r$) can be covered with $\left\lceil \frac{r'}{3} \right\rceil$ 3-modems, as the induction assumption.

Now, suppose that P is an orthogonal polygon with r reflex vertices. We are going to show that it can be covered with $\left\lceil \frac{r}{3} \right\rceil$ 3-modems.

We consider a reflex vertex of the polygon P and continue one of its incident edges to the interior of the polygon so that it would hit boundary at the first time. So, two orthogonal sub polygons with r_1 and r_2 reflex vertices will be obtained.

These polygons are called P_1 and P_2, respectively. It is clear that both of P_1 and P_2 are orthogonal polygons. Therefore we apply the induction assumption on them. It is obvious that $r = r_1 + r_2 + 1$. Some cases occur as follows:

- There exist the integers k_1 and k_2 such that $r_1 = 3k_1$ and $r_2 = 3k_2$. Based on the induction assumption, $\left\lceil \frac{r_1}{3} \right\rceil = k_1$ and $\left\lceil \frac{r_2}{3} \right\rceil = k_2$ are the numbers of 3-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 3-modems that are sufficient for covering P is $k_1 + k_2$. We want to show that $k_1 + k_2 \le \left\lceil \frac{r}{3} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{3} \right\rceil = \left\lceil \frac{r_1+r_2+1}{3} \right\rceil = \left\lceil \frac{3k_1+3k_2+1}{3} \right\rceil = k_1 + k_2 + 1 > k1 + k2$. Since the polygon P can be covered with $k_1 + k_2$3-modems, it can be covered with $\left\lceil \frac{r}{3} \right\rceil$ 3-modems.
- There exist the integers k_1 and k_2 such that $r_1 = 3k_1$ and $r_2 = 3k_2 + 1$. Based on the induction assumption, $\left\lceil \frac{r_1}{3} \right\rceil = k_1$ and $\left\lceil \frac{r_2}{3} \right\rceil = k_2 + 1$ are the numbers of 3-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 3-modems that are sufficient for covering P is $k_1 + k_2 + 1$. We want to show that $k_1 + k_2 + 1 = \left\lceil \frac{r}{3} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{3} \right\rceil = \left\lceil \frac{r_1+r_2+1}{3} \right\rceil = \left\lceil \frac{3k_1+3k_2+2}{3} \right\rceil = k_1 + k_2 + 1$. So, the polygon P can be covered with $\left\lceil \frac{r}{3} \right\rceil$ 3-modems.
- There exist the integers k_1 and k_2 such that $r_1 = 3k_1$ and $r_2 = 3k_2 + 2$. Based on the induction assumption, $\left\lceil \frac{r_1}{3} \right\rceil = k_1$ and $\left\lceil \frac{r_2}{3} \right\rceil = k_2 + 1$ are the numbers of 3-modems that are sufficient for covering P_1 and P_2, respectively. So, the number of 3-modems that are sufficient for covering P is $k_1 + k_2 + 1$. We want to show that $k_1 + k_2 + 1 = \left\lceil \frac{r}{3} \right\rceil$. Based on the substitution, we have $\left\lceil \frac{r}{3} \right\rceil = \left\lceil \frac{r_1+r_2+1}{3} \right\rceil = \left\lceil \frac{3k_1+3k_2+3}{3} \right\rceil = k_1 + k_2 + 1$. So, the polygon P can be covered with $\left\lceil \frac{r}{3} \right\rceil$ 3-modems.
- The reasoning about the other cases is similar to the above. ■

4. The k-Modem Covering the Plane in the Presence of a Monotone Orthogonal Polygons

In this section, the problem of covering the whole plane in the presence of a monotone orthogonal polygon by using modems is considered. The purpose is finding lower bounds on the power of a single modem to cover the whole plane.

Lemma 5. If P is an orthogonal polygon and L is a vertical line, then there is at least one point on L that any ray from this point crosses at most one vertical edge lying to the left of L and at most one vertical edge lying to the right of L.

Proof. We consider any line connecting two vertices which not lie on a same vertical line and both of them lie to the left or right of L. Then, we consider the topmost intersection point of these lines and L denoted by x. If y is a point on L higher than x, any ray from x crosses at most one vertical edge lying to the left of x because if a ray from y crosses two vertical edge lying to the left of L, the intersection point of L and the line connecting the lower endpoint of one of them and upper endpoint of the other is higher than y which is a contradiction. In the same way, it can be proved that any ray from y crosses at most one vertical edge lying to the right of L. ■

Theorem 5. The whole plane in the presence of a monotone orthogonal polygon can be covered with a 4-modem. This bound on the power of the modem covering the plane with a single modem is tight.

Proof. Let P be an x-monotone orthogonal polygon on n vertices. We assume L is a vertical line. By Lemma 9, there is a point on L denoted by x, such that any ray from this point crosses at most one vertical edge lying to the left of L and at most one vertical edge lying to the right of L. Suppose that we trace the boundary of the polygon. Two horizontal edges are said to be neighbor if there is only a vertical edge between them during tracing the boundary of P.

We assume that there is a ray from x such that crosses three horizontal edges a_1, a_2, and a_3 simultaneously which the pair a_1 and a_2, and the pair a_2 and a_3 are neighbors during tracing the boundary of P. Without loss of generality, we suppose that a_1, a_2, and a_3 lie to the right of the vertical line L. Likewise, we assume that tracing is left to right during the edge a_1. If tracing is right to left during the edge a_2, tracing should be right to left during the edge a_3 because of X-monotonicity of P.

It is obvious that no rays from the point x can cross a_2 and a_3 simultaneously which is contradiction with the assumption. Therefore, tracing should be left to right during the edge a_2. If tracing is left to right during the edge a_3, the ray from x which crosses three horizontal edges a_1, a_2, and a_3 should also cross two vertical edges simultaneously. It is contradiction because x is point such that any ray from x crosses at most one vertical edge at the same time (Lemma 9). So, tracing is right to left during the edge a_3. Consider the line connecting the right endpoints of a_1 and a_3 Since these point lie on the top of the ray from x which crosses the edges a_1, a_2, and a_3 simultaneously, the intersection point of the line connecting these endpoints and the line L is upper than the point x. That is contradiction because of Lemma 9.

So, any ray from x crosses at most two neighbor horizontal edges. Since P is an X-monotone orthogonal polygon, any ray from x can cross at most one other horizontal edge which is not neighbor of the horizontal edges that the ray crosses. So, any ray from x can cross at most three horizontal edges and one vertical edge. Therefore, a 4-modem placed at the point x covers the whole plane in presence of a monotone orthogonal polygon.

Figure 2(d) is a monotone orthogonal polygon. If the goal is covering the whole plane in presence of such that monotone orthogonal polygon with a single modem, the power 4 is necessary for the modem. So, the bound 4 on the power of the modem to cover the plane in presence of a monotone orthogonal polygon with a single modem is tight.■

5. Conclusions

In this paper, we study the k-modem art gallery problem and the problem of covering the plane for some special families of the polygons. It seems that these problems for the general polygons is rather challenging. So, the main open problem remaining is solving these problems for the general polygons.

References

[1] O. Aichholzer, R. Fabila-Monroy, D. Flores-Pealoza, T. Hackl, C. Huemer, J. Urrutia, and B. Vogtenhuber, "Modem illumination of monotone polygons," *Proc, European Workshop on Computational Geometry*, pp. 167-170, 2009.

[2] B. Ballinger, N. Benbernou, P. Bose, M. Damian, E. D. Demaine, V. Dujmovi, R. Flatland, F. Hurtadoy, J. Iacono, A. Lubiw, P. Morin, V. Sacristany, D. Souvaine, and R. Uehara, "Coverage with k- transmitters in the presence of obstacles," *Proc, Int'l conf. Combinatorial optimization and applications*, pp. 1-15, 2010.

[3] V. Chv'atal, "A combinatorial theorem in plane geometry," *Journal of Combinatorial Theory*, vol. B, no. 18, pp. 39-41, 1975.

[4] R. Fabila-Monroy, A. Vargas, and J. Urrutia, "On modem illumination problems," *Proc, Encuentro s de Geometria Computacional*, pp. 1-11, 2009.

[5] J. Urrutia, *Art Gallery and Illumination Problems:* Elsevier Science Publisher, 2000.

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