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Orthogonal Thickness of Graphs

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Abstract

In this paper, we introduce the concept of orthogonal thickness of graph. Orthogonal thickness of a graph G is the minimum number of planar graphs with maximum degree less than or equal to 4 whose union is G. We compute lower and upper bounds for orthogonal thickness and show that they are tight.

Next, we introduce the concept of layered orthogonal drawing and develop an algorithm for layered orthogonal drawing of K_n in $\lceil n/4 \rceil$ layers. Then, we extend the results to develop an algorithm for three-dimensional orthogonal (line-) drawing of K_n in a $2n \times 2n \times \lceil n/4 \rceil$ box. This drawing has at most 2 bends per edge and a total number of $n(n-2)$ bends.

Keywords: Orthogonal Drawing, Three-Dimensional Orthogonal Drawing, Thickness, Geometric Thickness.

1. Introduction

Orthogonal drawings have numerous practical applications in circuit schematics, data flow diagrams, entity relationship diagrams, etc. A two-dimensional orthogonal drawing is a drawing of a Graph in which every vertex is represented by a point and every edge is represented by a chain of horizontal and vertical segments. The place where the direction of an edge is changed, is called a bend. One of the most important criteria in an orthogonal drawing is minimizing the number of bends.

In this paper we consider *planar* orthogonal drawing, i.e. orthogonal drawings where no two edges cross except at a common end vertex. A planar Graph with maximum degree less than or equal to k is called k-planar. Orthogonal drawings in three dimensions are defined similarly, in these drawings each vertex is represented by a point and each edge is represented by a chain of segments that are parallel to one of the x-, y-, or z-axis [2, 3]. In this type of orthogonal drawing the degree of each vertex must be less than or equal to 6. However, there are other ways to draw a General Graph in an orthogonal way, one of them is orthogonal (line-) drawing that is a three-dimensional orthogonal drawing where each vertex is represented by an axis parallel line segment. Orthogonal drawings have direct applications in VLSI design and layout. In VLSI layout, a planar orthogonal drawing of a Given Graph is embedded in a plane.

Orthogonal drawing has three disadvantages when applying to VLSI layouts:

1. The degree of each vertex is restricted to 4 (in two-dimensional drawings) or 6 (in three-dimensional drawings). In orthogonal drawing of Graphs with higher vertex degrees, the vertices are represented by boxes, line segments or cycles [8, 3]. This increases the number of bends and the area (volume) of the drawing.
2. Non planar Graphs do not have planar orthogonal drawings in 2-D plane.
3. VLSI technology limits circuits to have few layers, it means that the size of one of the dimensions in the drawing must be restricted; but this condition does not hold in most three-dimensional drawing approaches [1].

In this paper, we introduce the concept of orthogonal thickness and compute tight lower and upper bounds for it. We also introduce the concept of layered orthogonal drawing and find such drawings for complete Graphs. Then we transform it to a three-dimensional orthogonal (line-) drawing for the Graph. The idea is to decompose the Graph to 4-planar subgraphs and find an orthogonal drawing of each subgraph such that the position of every vertex is fixed in the drawing of all subgraphs. In this type of orthogonal drawing we develop a way to overcome the disadvantages mentioned above: we can draw Graphs with high vertex degree and we can also restrict the size of one of the dimensions of the final drawing. The concept is a related to thickness, orthogonal drawing and Geometric thickness of Graphs [7, 13].

This paper is organized as follows: Basic definitions and results are in Section 2. The definition of orthogonal thickness and upper and lower bounds for it are presented in Section 3. In Section 4 orthogonal layered drawing is introduced and an algorithm for layered orthogonal drawing of K_n is developed. Section 5 is devoted to constructing a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing. Results and conclusions are summarized in Section.

2. Preliminaries

We need some definitions and results of Graph theory. We assume the definitions presented in [4]. Let $G=(V,E)$ be a Graph and $U\subseteq E$. The subgraph of whose vertex set is the set of end vertices of edGes in U and whose edge set is, is called the subgraph of induced by and is denoted by $G[U]$.

Planar Graphs have a natural advantage for visualization and some Graph problems can easily be solved when restricted to planar Graphs [8]. For non-planar Graphs, it is a natural question how far they are from being planar. There are different ways to answer this question. A known way is to determine the thickness of a Graph. The thickness of a Graph *G*, denoted by $\theta(G)$ is the minimum number of planar subgraphs of whose union is. There are two Good surveys on the topic by Liebers [9] and Mutzel et.al [10].

In an orthogonal drawing, the place where an edge changes its direction is a bend. An important criterion in orthogonal drawing is having a small number of bends. Having small area in two-dimensional drawings, or small volume, in three-dimensional drawings is another important criterion in orthogonal drawings. The research on the topic is extensive [8, 11].

3. Orthogonal Thickness

Let G be a Graph with the edge set E. An orthogonal layering of G is a decomposition of E into subsets $E_1,E_2,\ldots,E_k$ such that for each $1\le i\le k, G[E_i]$ is 4-planar. For is called an orthogonal layer of G. The minimum number of orthogonal layers of G is called the orthogonal thickness of G and is represented by $\hat{\theta}(G)$ [13].

In fact, orthogonal thickness of a Graph is a constrained version of thickness and for a Graph *G* the difference between $\theta(G)$ and $\hat{\theta}(G)$ can be very large. A simple example is $K_{1,n}$; it is clear that $\theta(K_{1,n})=1$ but $\hat{\theta}(K_{1,n})=\lceil n/4\rceil$.

3.1. Lower Bound on Orthogonal Thickness of Graphs

By definition of orthogonal thickness, $\hat{\theta}(G)=1$ if and only if G is 4-planar. In this section we present a lower bound on orthogonal thickness of Graphs. Since each vertex has degree at most 4 in each layer, we can obtain a lower bound on orthogonal thickness of Graphs.

Property 1: [13] Let *G* be a Graph of maximum vertex degree Δ . then $\hat{\theta}(G)\ge\lceil\Delta/4\rceil$.

We next prove that this bound is tight. We use the following lemma to prove this claim.

Theorem 1: Let be a planar Graph of maximum degree. then.

Proof: It is proved [6] that every planar Graph of maximum degree $\Delta\ge 8$ is Δ -edge colorable. Now Let G be a Graph with this condition and find a Δ -edge coloring of it. Now, let G_i be the subgraph induced by all edges of color $4i+1,\ldots,4i+4$, for $i=1,\ldots,\lceil\Delta/4\rceil$. The subgraphs G_i are 4-planar and this proves the theorem. □

3.2. Upper Bound on Orthogonal Thickness of Graphs

In this section we compute the orthogonal thickness of complete Graphs. This introduces and upper bound on orthogonal thickness of Graphs.

The problem of determining the orthogonal thickness of a Graph is, in other words, determining the thickness of the Graph when the degree of vertices in every planar subgraph is constrained to be less than or equal to 4. This problem is studied by Bose and prabhu [5]. They studied the problem under the name thickness of complete Graphs with degree constrained vertices. In this section we present a similar argument to find the orthogonal thickness of K_n .

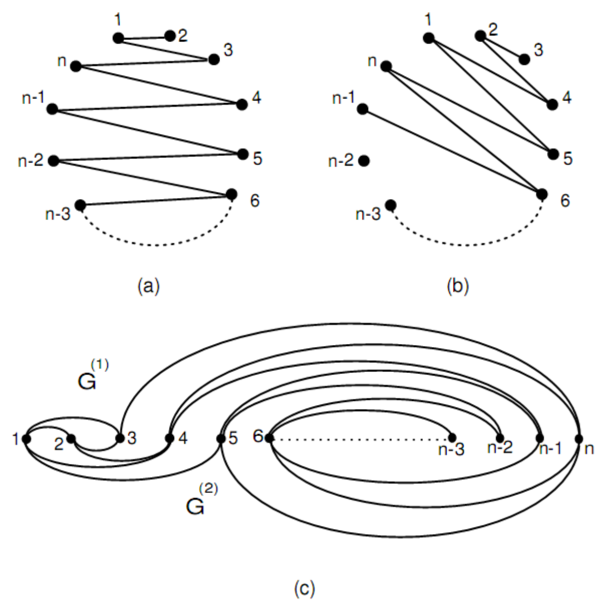

Figure 1. The Graphs (a): $G^{(1)}$ (b): $G^{(2)}$ and (c): G_1

Let $n = 4p, p \in IN$ and $\{1,2,...,n\}$ be the vertex set of $G = K_n$. Suppose that the vertices of *G* are placed on a circle in the clockwise order. For $1 \le k \le n$, define the edge set $E^{(k)} = \{ij : i + j \equiv_n 2k + 1 \text{ or } i + j \equiv_n 2k + 2\}$ and $G^{(k)} = G[E^{(k)}]$. Figure 1-(a) shows $G^{(1)}$. One can construct $G^{(i)}$ recursively by 1 unit clockwise shifting of vertices of $G^{(i-1)}$ around the circle. Figure 1-(b) shows $G^{(2)}$. For $1 \le k \le n/4$, $G^{(k)}$ is a spanning path in G with end vertices *k+1* and $n/2 + k + 1$, So $G^{(k)}$ is planar.

Now, for $1 \le k \le n/4$, let $G_k = G[E^{(2k)} \cup E^{(2k-1)}]$. Figure 1-(c) shows a planar drawin G of G_1.

Lemma 1: G_k is 4-planar.

Proof: For $1 \le k \le n/2$, $G^{(2k)}$ and $G^{(2k-1)}$ are two spanning paths with end vertices $2k+1, 2k, 2k + n/2 + 1$ and $n/2 + 2k$. So, the degree of these vertices in G_k is 3 and the degree of all other vertices is 4.

On the other hand, there exists a planar drawing for G_k as follows: Just arrange the vertices 1, 2, …, *n* on a line and draw the edges of $G^{(2k-1)}$ on one side of the line and the edges of $G^{(2k)}$ on the other side. Since $G^{(2k-1)}$ and $G^{(2k)}$ are planar, the edges can be drawn in a way that no crossing occurs between the edges on each side of the line. Figure 1 shows a planar drawing of G_1. So G_k is 4-planar. □

Theorem 2: Let $n = 4p$ where *p* is an integer. Then $G_1, G_2, ..., G_{n/4}$ form a decomposition of K_n to 4-planar subgraphs.

Proof: Lemma 2 proves that for $1 \le k \le n/2$, G_k is 4-planar. Now we prove that every edge of K_n is in exactly one G_k. Let $e = ij$ be an edge of K_n and $i + j \equiv_n r$, where $0 \le r \le n-1$. Now if *r* is even, then $e \in E^{(\frac{r-2}{2})}$, and if *r* is odd, then $e \in E^{(\frac{r-1}{2})}$. So, every edge of K_n is in exactly one of $G_1, G_2, ..., G_{n/4}$. By Lemma 2, these Graphs are 4-planar. So, $G_1, G_2, ..., G_{n/4}$ form a decomposition of K_n to 4-planar subgraphs.□

Theorem 2 shows that for n=4p, $\hat{\theta}(K_n) = n/4$. In the case that $n = 4p + r$, $0 \le r \le 3$, we can use the above argument for K_{n-r+4}. Note that in this case, $\lceil (n + r - 4)/4 \rceil = \lceil n/4 \rceil$. Thus we have the following result:

Theorem 3: For $n \ne 4p + 1$, $\hat{\theta}(K_n) = \lceil n/4 \rceil$.

Proof: Since the maximum degree of K_n is *n*-1, using Property 1 and the fact that for $n \ne 4p + 1$, $\lceil (n-1)/4 \rceil = \lceil n/4 \rceil$, the result is immediate. □

Bose and Prabhu [5] proved that $\hat{\theta}(G) = \lfloor (n+3)/4 \rfloor$ except possibly when $n = 4p + 1$ for p=3, 4, 5, …. Their result is equal to theorem 2. Also for $n = 4p + 1$, our bound equals their bound.

We can summarize the result of arguments in sections 3.1 and 3.2 to Get tight upper and lower bounds on orthogonal thickness.

Theorem 4: Let *G* be a Graph with *n* vertices and maximum vertex degree Δ. Then $\left\lceil \frac{\Delta}{4} \right\rceil \le \hat{\theta}(G) \le \left\lceil \frac{n}{4} \right\rceil$. □

4. Layered Orthogonal Drawing

Let G be a Graph and $G_1, G_2, ..., G_k$ be the orthogonal layers of G. A drawing of G composed of two-dimensional orthogonal drawings of the Graphs $G_1, G_2, ..., G_k$ in *k* distinct planes where each vertex $v \in V(G)$ has the same *x*- and y-coordinate in the drawing of $G_1, G_2, ..., G_k$ is called a *layered* orthogonal drawing of G with *k* layers. Figure 2 shows a layered orthogonal drawing. In this section, we show that there exists an orthogonal layered drawing for each Graph and then compute a layered orthogonal drawing of K_n with $\lceil n/4 \rceil$ layers.

In order to show that there is a layered orthogonal drawing for each orthogonal layering, we use the following lemma.

Lemma 2 [12]: Each planar Graph with *n* vertices admits a planar drawing that maps each vertex to a pre specified distinct location in plane and each edge to a polygonal curve with O(n) bends. More over such a drawing can be constructed in $O(n^2)$ time .

The following theorem proves the existence of layered orthogonal drawing for every Graph with every orthogonal layering.

Theorem 5: Let G be a Graph with orthogonal layers $G_1, G_2, ..., G_k$. Then G has a layered orthogonal drawing with these layers.

Proof: Each subgraph $G_1, G_2, ..., G_k$ is 4-planar, so it has an orthogonal drawing in the plane. We construct a layered orthogonal drawing Γ in an inductive manner. G_1 has an orthogonal drawing Γ_1. Let $P_1, P_2, ..., P_k$ be the location of vertices of G_1. By lemma 2, G_2 has a polyline drawing where its vertices are at fixed points $P_1, P_2, ..., P_k$. We can transform the polyline drawing to an orthogonal drawing, and after resizing the Grid and Γ_1, we would Get a layered orthogonal drawing of G_1 and G_2. We can continue and construct a drawing for G in $k-1$ steps. □

The drawings constructed in above theorem, might have a large number of bends and the area of each layer might be very large. In next section we construct a layered orthogonal drawing for K_n that has a relatively small number of bends and area.

4.1. Layered Orthogonal Drawing of K_n

Using the same argument as in section 3-2, we can assume that $n = 4p$. Let $V = \{1,2,...,n\}$ be the vertex set of K_n. For

$i = 1,2,...,n$, let the vertex i have coordinate $(2i-1, 2i-1)$ in the xy-plane. The straight line segment passing all of these points decomposes the $2n \times 2n$ square to two triangles: one triangle on the top-left and the other on the bottom-right.

Remember that $G_k = G[E^{(2k)} \cup E^{(2k-1)}]$ and for $1 \le k \le n/4$, $G[E^{(2k)}]$ and $G[E^{(2k-1)}]$ are two paths. We draw the edges of $G[E^{(2k-1)}]$ in the top-left triangle and the edges of $G[E^{(2k)}]$ in the other triangle.

Start drawing of $G[E^{(2k)}]$ from the vertex $2k-1$ and the edge connecting it to $2k$. It can be drawn in the right down triangle with only one bend, it first Goes left and then down. The next edge that connects $2k$ and $2k+1$, can be drawn with two bends, the edge route is left, up and then right (see Figure 2) All other edges of $G[E^{(2k)}]$, except the last edge, can be drawn with two bends and the last edge can be drawn with one bend. The edges in $G[E^{(2k-1)}]$ can be drawn in the same way in the bottom-right triangle. Figure 2 shows the orthogonal drawing of G_1. So we have the result. □

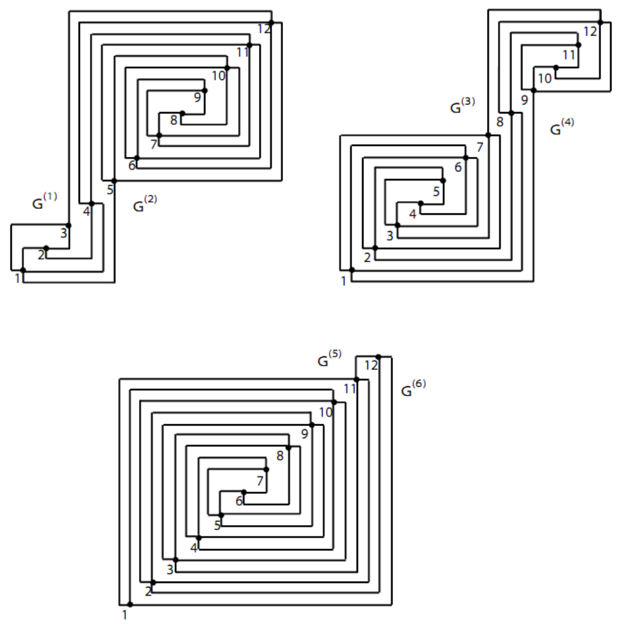

Figure 2. OrthoGonal layered drawinG of K_{12}. The three layers are drawn in separate planes

Theorem 6: The Graph K_n has a layered orthogonal drawing with $\lceil n/4 \rceil$ layers such that each layer is drawn in a $2n \times 2n$ square with $4n-8$ bends.

Proof: For $1 \le k \le \lceil n/4 \rceil$, It is easy to see that the drawings of G_k is planar and orthogonal. Also, the vertices have fixed coordinates in all drawings. Hence, the drawing is a layered orthogonal drawing.

The area of the orthogonal drawing of each orthogonal layer is $2n \times 2n$, since the vertices are placed in a $2(n-1) \times 2(n-1)$ square and the edges are drawn in a $2n \times 2n$ square. There are at most $2n-2$ edges in each layer, four of them have one bend and the other edges have two bends, so the total number of bends in the drawing of each layer is $4n-8$. □

Figure 2 shows a layered orthogonal drawing of K_{12}.

4.2. Construction of a Three-Dimensional Orthogonal (Line-) Drawing

In this section we construct a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing. Let G be a Graph, $G_1, G_2, ..., G_t$ be orthogonal layers of G and Γ be an orthogonal layered drawing of G with these layers. For $1 \le k \le t$, suppose that Γ_k is the restriction of Γ to the edges of G_k.

Construct a three-dimensional orthogonal (line-) drawing $\hat{\Gamma}$ of G in this way: For each point (x, y) in Γ_k, let (x, y, k) be a point in $\hat{\Gamma}$ and for each vertex *v* of G with coordinate (a,b) in Γ_1, represent *v* by a *z*-axis parallel segment with end points $(a,b,1)$ and (a,b,t). Figure 3 shows a layered orthogonal drawing and the related three-dimensional orthogonal drawing.

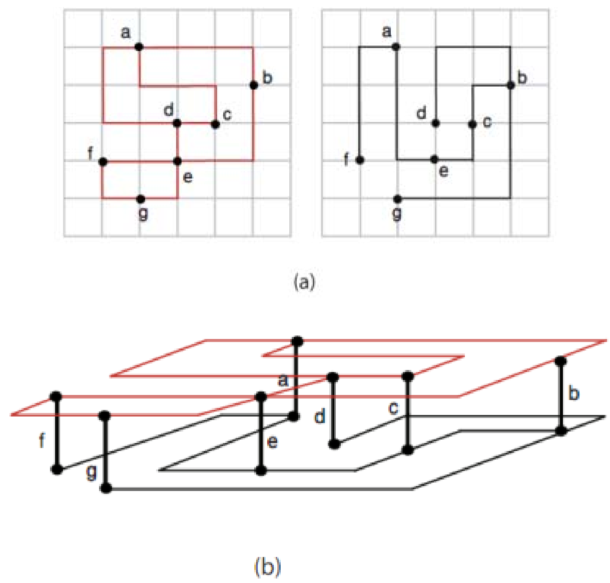

Figure 3. A layered orthoGonal drawinG, (a) The two orthoGonal layers are shown separately, b) the three-dimensional orthoGonal (line-) drawinG of the layered orthoGonal drawinG. Vertices are shown by thick line seGments

Now we can translate Theorem 3 for three-dimensional orthogonal (line-) drawings:

Corollary 1: The Graph K_n has a three-dimensional orthogonal (line-) drawing in a $2n \times 2n \times \lceil n/2 \rceil$ box with at most $n(n-2)$ bends.

Corollary 1 proves that there is a drawing of K_n with volume $n^3 - 4n^2$, at most 2 bends per edge and the total number of $n(n-2)$ bends.

There are other bounds on the volume of orthogonal drawing of K_n. Biedl et al. [2] proved that there is an orthogonal drawing of K_n with volume $O(n^{2.5})$ and at most three bends in each edge. In the same paper, Biedl et al. introduced a three-dimensional orthogonal (line-) drawing of K_n with at most two bends in each edge and with volume $O(n^3)$. In fact their drawing is bounded in the $n/2 \times n/2 \times n/2$ box. The advantage of our drawing to this drawing is that in our drawing all vertices are represented by z-axes parallel segments. Since there are limitations on the number of layers in a VLSI layout, $n/2$ layers is very hard to implement. But the drawing presented in this paper can easily be implemented in these circuits by using several parallel VLSI layers.

As an application of layered orthogonal drawing in VLSI layout, suppose that the size of the final orthogonal drawing of a three-dimensional orthogonal drawing is bounded in two dimensions. The question is if it is possible to find a three-dimensional orthogonal (line-) drawing with this restriction, and if so, find such a drawing with the smallest size of the third dimension.

Our result in this paper implies that if the drawing of K_n is restricted to have the size of two dimensions equal to $2n$, the size of the other dimension must be at least $\lceil n/4 \rceil$ and at most two bends are needed in each edge.

5. Conclusions and Open Problems

In this paper we introduced concepts of orthogonal thickness and layered orthogonal drawing of Graphs. We computed tight lower and upper bound for orthogonal thickness of a Graph by computing it for 2^p-planar Graphs and complete Graphs. Then, we presented an algorithm for constructing a layered orthogonal drawing of complete Graphs. We also constructed a three-dimensional orthogonal (line-) drawing from a layered orthogonal drawing.

The concept of orthogonal thickness is a special case of degree constrained thickness. An important part of the problem is developing algorithms for orthogonal layered drawing and the resulting three-dimensional orthogonal (line) drawing. In this drawing, vertices are the only *z*-axis parallel parts of the drawing. We believe that the decomposition of edges and their special layout make this type of orthogonal (line-) drawing more applicable than the existing three-dimensional orthogonal drawing styles, since in constructing the physical model of the drawing, it is enough to construct each layer separately. Then locate the place where each vertex passes one of the layers, and then insert a *z*-axis parallel segment (or connector) that passes all layers.

On the other hand, this method can be applied when we want to keep the size of one or more of the dimensions small. Existing algorithms for three-dimensional orthogonal drawings try to keep the volume of the drawing small, so the drawing that they produce is much like a cube [2]. In our method the size of one of the dimensions is as small as possible. In VLSI technology, there is a limitation on the number of layers.

We computed the orthogonal thickness and layered orthogonal drawing for complete Graphs. The bounds in this type of drawing are upper bounds on the number of bends and volume in three-dimensional orthogonal drawings. But the problem of computing the orthogonal thickness for other non-planar Graphs is still interesting. And there are some open problems:

1- What is the complexity of computing orthogonal thickness of a Given Graph?

2- Given two 4-planar Graphs on the same vertex set, how can one construct a layered orthogonal drawing with these two layers?

3- For an integer *k*, whether $\hat{\theta}(K_{4k+1})$ equals *k* or *k*+1?

Is there any class of forbidden subgraphs that if the Graph contains one of them as a minor, then the orthogonal thickness of the Graph would be more than two?

References

[1] T. Biedl, J. R. Johansen, T. Shermer, and D. R. Wood, "Orthogonal Drawings with Few Layers," *Proc, Int'l Conf. Graph DrawinG Contest*, pp. 297-311, 2002.

[2] T. Biedl, T. Shermer, S. Whitesides, and S. Wismath, "Bounds for Orthogonal 3-D Graph Drawing," *Journal of Graph Algorithms and Applications*, vol. 3, no. 4, pp. 63-79, 1999.

[3] T. Biedl, T. Thiele, and D. R. Wood, "Three-Dimensional Orthogonal Graph Drawing with Optimal Volume," *Journal of AlGorithmica*, vol. 44, no. 3, pp. 233-255, 2006.

[4] J. A. Bondy, and U. S. R Murthy, *Graph theory with applications*, Elsevier North-Holland, 1976.

[5] N. K. Bose, and K. A. Prabhu, "Thickness of complete Graphs with degree constrained vertices," *IEEE Trans. Circuit and Systems*, vol. 24, no. 4, pp. 184-190, 1977.

[6] M. Chrobak, and T. Nishizeki, "Improved edge-coloring algorithms for planar Graphs," *Journal of Algorithms,* vol. 11, no. 1, pp. 102-116, 1990.

[7] M. B. Dillencourt, D. Eppstein, and D. S. Hirschberg, "Geometric thickness of complete Graphs," *Journal of Graph Algorithms and Applications*, vol. 2, no. 39, pp. 5-17, 2004.

[8] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis, *Graph drawing, algorithms for visualization of Graphs*, Prentice Hall, 1999.

[9] A. Liebers, "PlanarizinG Graphs- A Survey and Annotated BiblioGraphy," *Journal of Graph, Algorithms and Applications,* vol. 5, no. 1, pp. 1-74, 2001.

[10] P. Mutzel, T. Odenthal, and M. Scharbrodt, "The thickness of Graphs: a survey," *Journal of Graphs and Combinatorics,* vol. 14, no. 1, pp. 59-73, 1998.

[11] T. Nishizeki, and M. D. Rahman, *Planar Graph drawing*, Lecture notes series on computing, world scientific, 2004.

[12] J. Pach, and R. WenGer, "EmbeddinG planar Graphs at fixed vertex locations," *Journal of Graphs and Combinatorics*," vol. 17, no. 4 , pp. 717-728, 2001.

[13] M. Tahmasbi, and S. M. Hashemi, "*Orthogonal thickness of Graphs*," proceedinG of 22nd conference on computational Geometry, (CCCG 2010), Winnipeg, Canada, Augest 2010.

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