Two Monetary Applications of Optimal Control

Reza Habibi

Department of Banking,
Iran Banking Institute, Central Bank of Iran
habibi1356@yahoo.com

Abstract. This manuscript first explores the application of optimal control theory in the formulation of arbitrage detection in exchange markets, addressing the approaches that price makers should take to derive the fair prices. The primary purpose of this study is to develop a robust optimal control model that enables price makers such as central banks to effectively manage macroeconomic factor exchange rate. We utilize methodologies that integrate stochastic control techniques and numerical simulations. In the second application, we apply the optimal control method to red-black game. To implement the game strategies, a Monte Carlo framework is advised.

Key Words: Arbitrage detection, Foreign exchange market, Monte Carlo, Optimal control, Red-Black game

JEL Classification: C71

1 Introduction. The optimal control is useful technique which has many applications in engineering, control theory, economics and finance. Its aim is to describe the value function and find a control whose cost or reward achieves the optimum value over all other controls. Optimal control theory is a mathematical framework used to find the best possible control strategies for dynamic systems over time. The optimal control is a powerful tool for improving decision-making processes. As financial markets grow more complex, the use of this method will likely become even more prevalent in achieving optimal outcomes. It provides a structured approach to dynamic systems, helping economists, investors, and policymakers develop strategies that respond effectively to changing economic conditions. In finance and especially in foreign exchange market management, it can be applied in various ways to improve decision-making and optimize outcomes. Some financial key applications of optimal control are dynamic portfolio optimization, wealth accumulation models, identifying hedging strategies, value-at-risk optimization, credit risk management, central bank policy rules, and *etc*, see references [2] and [3], for more details.
Optimal control in stochastic games also involves making decisions in a setting where multiple players interact, and the outcomes are influenced by random events. The primary goal is to develop strategies that maximize (or minimize) expected payoffs over time while considering both the actions of other players and the inherent uncertainties in the environment, see [12, 17].
Optimal control theory has been increasingly applied in the foreign exchange market to enhance trading strategies, manage risks, and optimize portfolio allocations. Some highlights key studies in this field are
-- Dynamic portfolio optimization. One of the primary applications of optimal control in the FX market is dynamic portfolio optimization. Researchers have utilized stochastic control methods to develop strategies that adjust currency holdings in response to market fluctuations. For instance, see [12].

-- Risk management. Optimal control has also been employed for risk management in FX trading. [7] explored the use of optimal control techniques to minimize Value-at-Risk (VaR) in currency trading. Their model adjusts the trading strategy based on the changing volatility of currency pairs, thereby improving risk-adjusted returns.
-- Algorithmic trading strategies. The integration of optimal control with machine learning algorithms has led to the development of sophisticated trading strategies. [6] demonstrated how reinforcement learning can be combined with optimal control frameworks to create adaptive trading systems that respond to market conditions in real-time, see [15, 18].
-- Market microstructure. Optimal control has been applied to study market microstructure effects in FX markets. [10] analyzed how optimal execution strategies can be derived using control theory to minimize market impact and transaction costs when executing large trades.
-- Macroeconomic factors. Lastly, optimal control models have been used to incorporate macroeconomic factors influencing currency values. [15] developed a model that integrates economic indicators into optimal control frameworks to forecast exchange rates more accurately, see [19, 20].

Optimal control theory and stochastic games are two significant areas in mathematical optimization and game theory. The intersection of these fields has garnered considerable attention, particularly in the context of dynamic decision-making under uncertainty. For example in robust control in stochastic games, see [8] who introduced robust control techniques that account for model uncertainties, providing a framework for designing strategies that perform well under a range of possible scenarios.

In multi-agent systems, see [1] who examined cooperative and competitive strategies among agents using optimal control principles, demonstrating how agents can learn and adapt their strategies in dynamic environments. Recent studies have focused on integrating machine learning techniques with optimal control in stochastic games. [20] proposed a framework that combines deep reinforcement learning with optimal control strategies, enabling agents to learn optimal policies in complex environments with high-dimensional state spaces.

This paper is organized as follows. In section 2, two applications of optimal control are given. They are controlling exchange rates in foreign exchange market and determining the amount of bets in red-black game, see [5, 9]. In section 3, some real data analyses in two applications are provided. Asymptotic and Monte Carlo based results are also given. This section contributes to the field by providing empirical evidence of the efficacy of optimal control methods in enhancing foreign exchange policy effectiveness and stability as well as into the stochastic games such as red-black games. Finally, conclusions are given in section 4.

2 Two applications. Here, the mathematical aspects of two applications of optimal control in arbitrage detection in foreign exchange market and red-black game stochastic game are given.

2.1 Arbitrage exchange rates. Arbitrage trading in foreign exchange involves finding price inefficiencies in different exchanges and exploiting them. Triangular arbitrage strategy involves using the three currency pairs and to purchase and to sale them, simultaneously, see [5,6]. Depending on their reserves, central banks may decide to buy foreign currency or sell the local currency in order to influence its value. In this way, they try to control the price of their currency in order to avoid either under or over-valuation.

a) Optimal control. Suppose x_t, u_t, y_t are the three exchange rates, for example, Yen to US (x_t) dollar, Yen to Euro (u_t) and Euro to US dollar (y_t), respectively. Naturally, to avoid the arbitrage opportunities, Japanese's monetary policy makers should make regulations about x_t, u_t in such a way that

$$x_t = u_t y_t.$$

Assume that policymakers have control over u_t but not over x_t. In practice, assuming x_t, y_t and u_t are integral-able, for every time $t \in [0, T]$, this equality cannot hold, and it is enough to minimize

$$\int_0^T (\mathrm{x}_t - \mathrm{u}_t \mathrm{y}_t)^2 dt$$

is minimized with respect to $\mathrm{u}_t, t \in [0, T]$. Here, the state variables are x_t, y_t and admissible control variable is u_t. Here, we assume that the above integral is finite.

Remark 1: (Executive summary). In this section, it is assumed that, the perfect no triangular arbitrage does not exist, in practice. Therefore the squared difference of $\mathrm{x}_t - \mathrm{u}_t \mathrm{y}_t$ is minimized. To make sure that $\mathrm{x}_t - \mathrm{u}_t \mathrm{y}_t$ is minimized for all time, the integral form of squared $\mathrm{x}_t - \mathrm{u}_t \mathrm{y}_t$ is studied. The method works well if maximum of absolute of errors $\mathrm{x}_t - \mathrm{u}_t \mathrm{y}_t$ is small.

b) Control solution. Suppose that x' is derivative of x with respect to time. It is assumed that, all exchange rates are differentiable with respect to time. It is enough, to assume that, they are smooth functions. In practice, x' is approximated, numerically. Notice that

$$x' = uy' + yu'.$$

Here, to use the Hamiltonian functional of optimal control, the following assumptions are supposed.

-all dynamics rates are differentiable. Exchange rates are smooth function of times.
-all integrals are finite
-there exists a unique solution for control exchange rate u
-There is no jump or discontinuity or extreme behavior in exchange rate processes, see [7, 8].

The Hamiltonian function is, see [7]

$$\mathrm{H} = (x - uy)^2 + \pi(uy' + yu'),$$

where π is the Lagrange coefficient. Notice that $\frac{\partial \mathrm{H}}{\partial u} = 0$ implies that

$$2y(x - uy) = \pi y'.$$

Next,

$$\frac{\partial \mathrm{H}}{\partial x} = -\pi'$$

implies that

$$\pi' = -2(x - uy) = -\pi \frac{y'}{y}.$$

Thus,

$$\frac{\pi'}{\pi} = -\frac{y'}{y}.$$

It is seen that $\pi = 1/y$. It is concluded that

$$u = \frac{x}{y} - \frac{y'}{2y^3}.$$

This is the optimal control u_{opt}, while $\frac{x}{y}$ is the actual controller, *i.e.*, u_{act}. That is,

$$u_{\text{opt}} - u_{\text{act}} = -\frac{y'}{2y^3}.$$

The following proposition summarizes the above discussion.

Proposition 1. Statements *a* and *b* are correct.

a) Policymakers have control over u_t, to minimize the effect of triangular arbitrage in a foreign exchange market, should choose the u_t such that

$$u = \frac{x}{y} - \frac{y'}{2y^3}.$$

b) The difference between optimal control u_{opt} and actual controller, *i.e.*, u_{act} is

$$u_{\text{opt}} - u_{\text{act}} = -\frac{y'}{2y^3}.$$

2.2 Red-Black game. From personal finance point of view, the optimal control is used to solve the stochastic game models. In particular, consider the situation where an individual may bet any integral amount not greater than his fortune and he will win his amount with specific probability. The red and black is an example. This game, which was first theoretically studied by [2], has been studied from various perspectives in the existing literature. Trivial types of games, and sub-fair, super-fair, fair games are well studied. The problem is extended to roulette and stochastic games see [17]. For comprehensive review in this field, see [18,19] and references therein. [9] presented their results in the form of a discrete time game and found that one of the optimal strategies is the bold play. [6] studied this game in the form of continuous time and found the optimal strategy of the game using the method of controlling the diffusion process. In this section, we follow a similar method.

a) Problem formulation. Suppose that one plays at

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

for some natural n's and let

$$x_k := x_{t_k} \in (0,1)$$

be the fortune of player at k-th game. Here, it is assumed that the betting size is

$$\delta_k = \delta(\frac{k}{n})$$

for some positive-value functions δ's. The J_k is dummy variable of player at k-th step, where

$$J_k = 1$$

if player wins at k-th game with probability of p_n and -1 if he loses with probability of $1 - p_n$. One can see that

$$\mu_n = E(J_k) = 2p_n - 1$$
$$\sigma_n^2 = 4p_n(1 - p_n).$$

Although, various types of red and black games are studied, in the literatures, however, the near unfair game is forgotten. Here, it is assumed that $p_n < 0.5$ and

$$p_n = 0.5(1 - \frac{\gamma}{\sqrt{n}})$$

for some positive parameters γ's. Here, it is assumed that p_n is far from 0 or 1.

b) Asymptotic results. In the case of near unfair game, it is easy to see that

$$\sqrt{n}\mu_n = -\gamma$$

and σ_n^2 goes to 1 as $n \to \infty$. Let

$$J_k^* = \frac{J_k - \mu_n}{\sigma_n}$$

be the standardized random variable and

$$s_k = s_n(k/n) = n^{-0.5} \sum_{r=1}^{k} J_k^*.$$

Using the Lyapunov condition for central limit theorems, (for this condition, see [4]) it is seen that $s_{[nt]}$ converges weakly in $D[0,1]$ to standard Brownian motion B_t. Notice that, variables in the summations are independent with finite second moments. Thus, the Lyapunov condition is satisfied.

Hereafter, the asymptotic behavior of fortune of player at k-th steps is studied. Really the player fortune x_k at the k-th step of game is written as stochastic integral with respect to empirical measure and the continuous mapping theorem implies that the sample stochastic integrals converges to population stochastic integral, because of strong law of large numbers.

Assuming the initial fortune of player is zero, k-th fortune is

$$x_k = \sum_{r=1}^{k} J_r \delta_r.$$

One can see that

$$n^{-\frac{1}{2}} x_n = \sigma_n n^{-\frac{1}{2}} \sum_{r=1}^{n} J_r^* \delta_r + \sqrt{n}\mu_n \frac{\sum_{r=1}^{n} \delta_r}{n}$$
$$= \int_0^1 \delta(t) d\, s_n(t) - \gamma \int_0^1 \delta(t) d\, \lambda_n(t)$$

where λ_n is the Lebesgue measure. As n gets large, then using the continuous mapping theorem, then $n^{-\frac{1}{2}} x_n$ converges to

$$\int_0^1 \delta(t) d\, B_t - \gamma \int_0^1 \delta(t) d\, t = \int_0^1 \delta(t) dB_t^*$$

where $B_t^* = B_t - \gamma t$ is the Brownian motion with drift $-\gamma$. Notice that

$$x_t = -\gamma \int_0^t \delta(s)ds + \int_0^t \delta(s)dB_s.$$

It is assumed that in practice $\delta(s)$ and B_s do not have common discontinuities, see [4]. Therefore, x_1 has normal

$$N(\mu_\delta, \sigma_\delta^2)$$

distribution where mean μ_δ and variance σ_δ^2 are given by

$$\mu_\delta = -\gamma \int_0^1 \delta(t)dt$$

$$\sigma_\delta^2 = \int_0^1 \delta^2(t)dt.$$

The following proposition summarizes the above discussion.

Proposition 2. The k-th fortune of red-black game player converges to

$$x_t = -\gamma \int_0^t \delta(s)ds + \int_0^t \delta(s)dB_s.$$

c) Monte Carlo framework. Here, the Monte Carlo method is applied to simulate path fortune

$$x_{t_1}, x_{t_2}, \dots, x_{t_k}.$$

Following notations of [9], the variable symbol in parentheses indicates the distribution of that variable. To simulate mentioned path, the bridge sampling technique is used. In fact, instead of sampling directly from the distributions of the variables, it is sampled from the following family of sequential conditional distributions.

$$(x_{t_1}|x_{t_k}), (x_{t_2}|x_{t_1}, x_{t_k}), \dots, (x_{t_{k-1}}|x_{t_{k-2}}, x_{t_k})$$

Here, x_{t_1} given x_{t_k} has normal distribution with mean

$$x_{t_k} + \gamma \int_{t_1}^{t_k} \delta_s ds$$

and variance

$$\int_{t_1}^{t_k} \delta_s^2 ds.$$

Here, it is assumed that all exchange rates have finite second moments, see [3]. For simulating conditional distribution

$$(x_t|x_s, x_u)$$

where $s < t < u$, consider two independent normal variables

$$X = x_t - x_s, Y = x_u - x_t,$$

and write this distribution as

$$x_s + (X|X + Y)$$

Let μ_x, μ_y, σ_x^2 and σ_y^2 be the means and variances of X, Y, respectively. Then, it is easy to see that

$$X|X + Y = z^*$$

has normal distribution with mean and variance given as below, respectively

$$\text{mean} = \mu_x + \frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}(\mu_x + \mu_y - z^*)$$

$$\text{variance} = \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}.$$

As $\delta = 1$, then

$$\mu_x = -\gamma(t - s),$$

$$\mu_y = -\gamma(u - t),$$

$$\sigma_x^2 = t - s$$

$$\sigma_y^2 = u - t.$$

Therefore, the mean and variance of

$$X|X + Y = z^*$$

is

$$(t - s)\left\{1 - \left(\gamma + \frac{z^*}{u-s}\right)\right\} =,$$

$$= \frac{(t-s)(u-t)}{u-s}.$$

The following proposition summarizes the above discussion.

Proposition 3. To simulate fortune path of player, it is enough to follow the conditional path

$$(x_{t_1}|x_{t_k}), (x_{t_2}|x_{t_1}, x_{t_k}), \dots, (x_{t_{k-1}}|x_{t_{k-2}}, x_{t_k})$$

where for $s < t < u$,

$$X|X+Y = z^*$$

has normal distribution with mean and variance, as above stated.

Algorithm 1.

i) Derive the conditional distribution

$$(x_{t_1}|x_{t_k}), (x_{t_2}|x_{t_1}, x_{t_k}), \dots, (x_{t_{k-1}}|x_{t_{k-2}}, x_{t_k})$$

Usually, these distribution are estimated by a fitting a regression equation and finding the distribution of errors of regression. If some normality test accepts the normality of consecutive conditional distribution, then the proposition 3 is applicable. If another distribution is accepted, that distribution should be identified with famous goodness of fit procedures in inferential statistics.

ii) Start sampling from $(x_{t_1}|x_{t_k})$*, and go ahead. It becomes a path of player' fortune.*

Notice that simulation method proposed here is Monte Carlo framework for simulating path of player's fortune which is used ion optimal control rule to find the optimal strategies. In Example 4 of next section, this method is applied.

d) Optimal rule. In a red-black game, mimicking [6], it is enough to optimize

$$\delta^{-1}(t),$$

however, here, the optimal control may be applied to find the optimizer function $\delta(t)$ which minimizes

$$E|x_1 - 1| \text{ or } E(x_1 - 1)^2$$

with respect to δ_t, where x_1 is the final fortune of player. According to the [4, 11], pages 21 and 44, respectively, it is seen that

$$|x_1 - 1| = x_1 + \int_0^1 \{2I(x_1 \le y) - 1\}dy,$$
$$E|x_1 - 1| = E(x_1) + \int_0^1 \{2\pi(y) - 1\}dy,$$

where

$$\pi(y) = P(x_1 \le y) = \Phi(\tfrac{y-\mu_\delta}{\sigma_\delta}),$$

where Φ is the distribution function of standard normal distribution. Notice that

$$E(x_1) = \mu = -\gamma \int_0^1 \delta(t)dt.$$

Therefore,

$$E|x_1 - 1| = 2\int_0^1 \Phi(\tfrac{y-\mu_\delta}{\sigma_\delta})dy - \gamma \int_0^1 \delta(t)dt - 1.$$

If we are interested to compute

$$E(x_1 - 1)^2$$

instead of

$$E|x_1 - 1|,$$

notice that

$$E(x_1 - 1)^2 = \sigma_\delta^2 + (\mu_\delta - 1)^2$$

The following proposition summarizes the above discussion.

Proposition 4. Notice that

$$E|x_1 - 1| = 2\int_0^1 \Phi(\tfrac{y-\mu_\delta}{\sigma_\delta})dy - \gamma \int_0^1 \delta(t)dt - 1.$$
$$E(x_1 - 1)^2 = \sigma_\delta^2 + (\mu_\delta - 1)^2$$

The optimum rule is proposed in simulation studies.

Remark 2: (Executive summary). The fortune of red-black game player is formulated by sum of product of betting amount δ_k and winning dummy variable J_k. Then, its asymptotic behavior is approximated by stochastic integrals with respect to Brownian motion. The Monte Carlo framework is presented to simulate small sample behavior of player fortune. Finally, optimal rule in red-black game is proposed.

3 Data analysis. Here, first, the applications of theoretical results of previous section are seen in the simulated data sets. The time span for all data sets is 15 December 2022 to 15 December 2023 (263 observations). Then, an economic implication is proposed.

3.1. Simulated examples. Here, some simulated examples are proposed.

Example 1: Euro to US. Consider the daily Euro to US dollar y_t. Time increment is $\frac{1}{264} = 0.0038$. Both $u_{opt}(t)$ and $u_{act}(t)$ are computed for , $t = 1, \dots, 263$. Then, their absolute values of errors are computed. The following table gives the descriptive summaries of errors. It is seen that errors are negligible and thus, the optimal control exchange rate works well. In this way, we can avoid the triangular arbitrage. The following Table gives summaries of

$$e_t = |u_{opt}(t) - u_{act}(t)|, t = 1, \dots, 263.$$

Table 1: Summaries of e_t

mean	stdev	max	min	Q1	Q2	Q3
0.408	0.332	1.098	0	0.158	0.338	0.583

Here, $Qi, i = 1,2,3$ are the first, second and third quartile of e_t's.

Example 2: Portfolio of currencies. Suppose that decision makers are going to control Yen to US dollar (x_t) by a relative price of Yen to a portfolio of currencies (u_t) (a portfolio of exchange rates which are controllable) Therefore, y_t is the relative price of given portfolio to US dollar. Suppose that

$$y_t = 0.5(\text{GBP/USD}) + 0.5(\text{CHF/USD}).$$

Again, the following Table gives the necessary summarizes:

Table 2: Summaries of e_t

mean	Stdev	max	min	Q1	Q2	Q3
0.344	0.289	1.05	0	0.134	0.275	0.483

It is seen that the method (optimal control for determining optimal exchange rate) works well for a portfolio of exchange rates.

Example 3: Monte Carlo. Here, a study on how changes in mixing portion of a portfolio model affect the results of Monte Carlo simulations is conducted. Consider a portfolio of exchange rate with

$$y_t = \alpha\left(\frac{\text{GBP}}{\text{USD}}\right) + (1-\alpha)(\text{CHF/USD}).$$

The following plot gives the mean of absolute errors e_t for $t = 1, \dots, 263$ versus $\alpha = 0.05\ (0.05) 0.95$. It is seen that the method works well-diversified portfolios. That is, as α closes to 0.5, the mean absolute of errors is less.

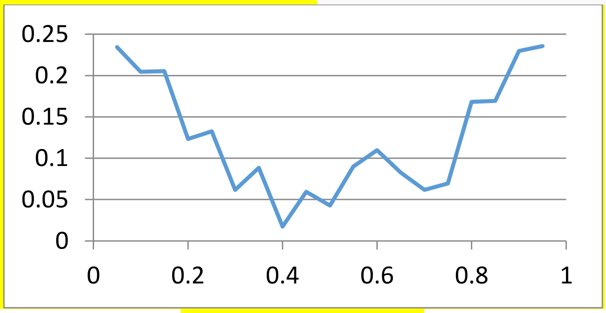

Fig. 1: Mean of $||e_t|$ vs. α

Applying the optimal control method proposed in the current paper to a portfolio of currencies will increase the reliability of the conclusions and allow for a better assessment of the stability of the applied method.

Example 4: red-black game. Let $\gamma = 0.1$ and $\delta(t) = 1$, then

$$\mu_\delta = -0.1 \text{ and } \sigma_\delta = 1.$$

Therefore,

$$E(x_1 - 1)^2 = 2.21$$

$$E|x_1 - 1| = 2\int_0^1 \Phi(y - 0.1)dy - 1.1.$$

Remark 3. The quantity

$$\int_0^1 \Phi(y - 0.1)dy = E(\Phi(U - 0.1))$$

where U is a uniform random variable. It is estimated using Monte Carlo simulation with 1000 repetitions. It is 0.655.

The optimal amount of betting is a switching rule

$$\delta(t) = \begin{cases} 1 & y_t > 0.1 \\ o & ow \end{cases}$$

Example 4: Simulation of player path. Let, again, $\gamma = 0.1$, $\mu_\delta = -0.1$ and $\sigma_\delta = 1$. The following plot gives different paths of fortunes player. The amount of bet is random variable with uniformly distributed on (0,1) law.

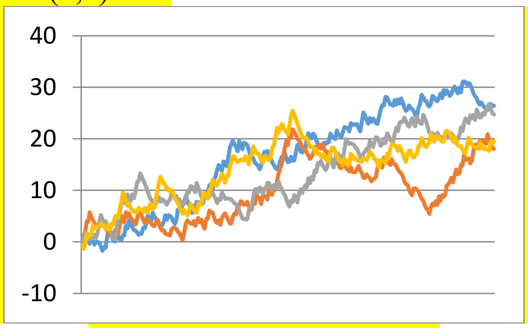

Fig. 2: Different paths of player's fortune

This figure shows the importance of optimal control of betting amount. It is seen that variability of path is too much.

Example 5: Euro to GBP. Consider the daily Great Britain Pound (GBP) to US dollar y_t. Time increment is $\frac{1}{264} = 0.0038$.

Table 3: Summaries of e_t

mean	Stdev	max	min	Q1	Q2	Q3
0.415	0.399	1.10	0	0.194	0.265	0.485

It is seen that the method works well for this exchange rate. However, the result is much better in Example 1 and for daily Euro to US dollar.

Example 6: Consider the case of example 1. Here, we incorporate a comparative analysis of optimal control (maximum principle) with alternative methods. The results of these methods are summarized in the following table. To obtain the simulation results for each method, we utilized ready-made MATLAB code rather than deriving a closed-form solution for each method. Based on these results, maximum principle method after reinforcement learning has the best results.

Table 4: Comparative analysis

Method	Advantage	$Max\|e_t\|$
Dynamic programming	Provides a systematic approach to finding optimal policies; can handle multi-stage decision processes	1.12
Pontryagin's maximum principle	Offers a way to derive optimal control laws directly from the system's dynamics and performance criteria.	1.05
Linear quadratic regulator	Provides a closed-form solution, making it computationally efficient and easy to implement.	1.15
Model predictive control	Can handle multivariable systems and constraints on inputs and states, providing a flexible control approach.	1.51
Value function approach	Provides a comprehensive framework for evaluating and comparing different control strategies	1.06
Calculus of variations	Provides a powerful mathematical tool for deriving optimal control laws.	1.21
Stochastic Optimal Control	Provides a framework for making optimal decisions in the presence of uncertainty.	1.35
Reinforcement learning	Capable of learning optimal policies without a predefined model of the system.	1.03

3.2 Economic implications. The economic implications of applying optimal control in arbitrage detection within exchange rates are significant and multifaceted:

-*Enhanced market efficiency*. Optimal control models can improve the detection of arbitrage opportunities, leading to quicker corrections in exchange rates. This reduces mis-pricings and fosters more efficient markets.

-*Reduced transaction costs and risks*. By accurately identifying arbitrage opportunities, traders can execute more effective strategies, minimizing unnecessary transaction costs and exposure to exchange rate risks.

-*Stabilization of exchange rates*. Implementing optimal control strategies can help central banks and financial institutions stabilize currency markets by preemptively addressing arbitrage-driven fluctuations.

-*Impact on currency valuation and policy*.

Better arbitrage detection influences the perceived fair value of currencies, informing monetary policy decisions and potentially reducing speculative attacks.

-*Promotion of fair trade and investment*. Accurate arbitrage detection ensures that exchange rates reflect true market conditions, encouraging fair international trade and attracting foreign investment.

-*Potential for market manipulation.* On the downside, sophisticated arbitrage detection might be exploited for speculative gains, potentially leading to increased volatility if not properly regulated.

-*Resource allocation efficiency*. Firms and investors can allocate resources more effectively when arbitrage opportunities are swiftly identified and acted upon, contributing to overall economic growth.

In summary, the integration of optimal control into arbitrage detection enhances the efficiency, stability, and fairness of currency markets, with broad implications for macroeconomic stability and international trade.

4 Conclusions. The application of optimal control techniques in stochastic games provides a robust framework for modeling and solving dynamic decision-making problems under uncertainty. By integrating stochastic processes with game-theoretic interactions, this approach allows for the analysis of optimal strategies in competitive or cooperative environments affected by randomness (e.g., noise, market fluctuations, or economical variability). Key contributions of this work include:

1) Theoretical insights: Derivation of conditions (e.g., Hamilton-Jacobi-Bellman equations or mean-field approximations) ensuring the existence and stability of exchange market as well as in betting in red-black stochastic games.

2) Computational methods: Development of numerical schemes to approximate solutions.

3) Practical applications: Demonstrated decision frameworks for personal finance, central bank's monetary policies, stochastic games, gambling, and *etc*, where agents adapt to uncertain environments.

4) Future directions: It may involve extending results to partially observable systems, incorporating machine learning for adaptive control, or addressing scalability in multi-agent settings.

5) Potential applications: This work underscores the potential of optimal control in stochastic games to unify dynamic optimization and strategic interaction for complex real-world systems.

In conclusion, the application of optimal control techniques to arbitrage detection within exchange rates offers a promising approach to enhancing market efficiency and stability. By providing a systematic framework for identifying and exploiting arbitrage opportunities in real-time, these methods can significantly reduce mis-pricings and mitigate excessive volatility driven by speculative activities. The findings underscore the potential for optimal control models to inform both traders and policymakers, leading to more accurate exchange rate assessments and improved monetary stability. Future research should focus on refining these models to incorporate market complexities and exploring their practical implementation in dynamic, real-world financial environments. Overall, this study highlights the valuable role of optimal control in advancing currency market analysis and fostering a more efficient and resilient global financial system.

References

[1] Baker C. "Multi-Agent Reinforcement Learning in Sequential Social Dilemmas." Proceedings of the 27th International Joint Conference on Artificial Intelligence. 2018. https://arxiv.org/abs/1702.03037

[2] Baxter M. "Financial Calculus: An Introduction to Derivative Pricing 3th *ed*," Cambridge University Press. 1996. DOI: https://doi.org/10.1017/S0266466698143050

[3] Bjork T. "Arbitrage Theory in Continuous Time 4th *ed*.,"Oxford University Press. 2009. DOI: https://doi.org/10.1093/0199271267.001.0001

[4] Borkar V. S. "Stochastic Approximation: A Dynamical Systems Viewpoint 4th ed.," Cambridge University Press. 2008. DOI: https://doi.org/10.1007/978-93-86279-38-5

[5] Bussiere M. and Fratzscher M. "Low Probability, High Impact: Policy Making and Extreme Events," Journal of Policy Model, vol. 30, no 1, pp. 111–121. 2016. https://doi.org/10.1016/j.jpolmod.2007.03.007

[6] Chen, L., Wang, Y., & Zhang, X. "Reinforcement Learning and Optimal Control for Algorithmic Trading in Foreign Exchange Markets." Journal of Computational Finance, 25(2), 45-67. 2021. https://arxiv.org/abs/1908.08036

[7] Deng, Y., Liu, H., & Zhang, J. "Optimal Control for Value-at-Risk Minimization in Foreign Exchange Trading." Quantitative Finance, 20(5), 789-802. 2020. https://www.ijfma.ir/article_23687_a640f3accaebcb81205e773a65e3d894.pdf

[8] Doyle, J. C. "Essentials of Robust Control". Prentice Hall. 1998. https: //www. amazon. com › Essentials-Robust-Control

[9] Fechner, W. "New Inequalities for Probability Functions in the Two-person Red-and-Black Game," Journal of Mathematical Analysis and Applications, vol. 491, no. 2, pp. 124-139. 2020. DOI: https://doi.org/10.48550/arXiv.2105.14263

[10] Foucault, T., Hombert, J., & Rosu, I. "Optimal Execution in Foreign Exchange Markets." Review of Financial Studies, 32(4), 1333-1365. 2019. https:// www. Researchgate .net › publication › 24790761.

[11] Glasserman P. and Kim K. K. "Beta Approximation for Bridge Sampling," Proceeding of the 2008 Winter Simulation Conference. Pp. 100-107. 2008.
DOI: https://doi.org/10.1109/WSC.2008.4736115

[12] Klein, P., & Bansal, R. "Dynamic Portfolio Optimization in Foreign Exchange Markets." Journal of Financial Economics, 132(3), 654-678. 2019. https://arxiv.org/abs/1812.11794

[13] Kurt B. and Ceritli T. Y. "A Bayesian Change Point Model for Detecting SIP-Based DDoS Attacks," Digital Signal Processing, vol. 77, no. 3, pp. 48–62. 2018. DOI. https://doi.org/10.1016/j.dsp.2017.10.009

[14] Lo Duca, M. and Peltonen, T. A. "Assessing Systemic Risks and Predicting Systemic Events," Journal of Banking Finance, vol. 37, no. 2, pp. 2183–2195. 2013. DOI. https://doi.org/10.1016/j.jbankfin.2012.06.010

[15] Menkhoff, L., Sarno, L., Schmeling, M., & Wagner, C. "Optimal Control and Macroeconomic Fundamentals in Foreign Exchange Markets." Journal of International Money and Finance, 70, 1-20. 2021. https://www.nature.com › nature-index › topics › dynamics

[16] Pestien V. C. and Sudderth W. D. "Continuous-Time Red and Black: How to Control A Diffusion to A Goal". Technical Report 426, pp. 89-95. Universities of Miami and Minnesota. 2013. DOI. https://doi.org/10.1287/moor.10.4.599
[17] Ross S. "Introduction to Stochastic Dynamic Programming 4th ed." Academic Press. 2012. DOI: https://doi.org/10.18261/9788215026718-2016
[18] Shiryaev, A. N. and Zhitlukhin M. V. "Optimal Stopping Problems. Part 2: Applications," Technical report. Steklov Mathematical Institute, Moscow, pp. 78-85. 2013. https://doi.org/10.1016/j.dsp.2017.10.009
[19] Tijms, H. "Stochastic Games and Dynamic Programming," Asia Pacific Mathematics Newsletter, vol.67, no.3, pp. 6-10. 2012. https://doi.org/10.3982/QE153
[20] Zhang, Y. "Deep Reinforcement Learning for Multi-agent Systems: A Review." IEEE Transactions on Cybernetics 65, 1-10. 2020. https://www.researchgate.net› publication › 34006846